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ON THE POSSIBILITY OF LOCAL BUCKLING OF THE SURFACE OF AN ELASTIC HALF-SPACE UNDER COMPRESSION

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The possibility of local buckling of the free surface of the lower half-plane under compression is studied in a static formulation within the framework of plane deformation. It is shown that in some media small subcritical strains can lead to local buckling of the half-plane surface. It is found that two forms of local surface buckling correspond to one critical compression load.

Key words: *local instability, elastic half-space, Fourier transformation.*

Biot [1] was the first to study the instability of the generally free surface of a half-plane for an incompressible medium. The instability of the generally free surface of the lower half-plane under compression was studied in [2]. The local axisymmetric buckling of the surface of an elastic half-space under compression was the subject of research in [3].

The present paper is concerned with studying the local buckling of the free surface of the lower half-plane under compression in a static formulation. Surface buckling is investigated assuming plane deformation with small homogeneous subcritical strains.

The elastic half-space is compressed along the Ox_1 axis by forces of intensity $\boldsymbol{p}$. The Ox_2 axis is perpendicular to the free surface.

The linearized equations of stability against the displacement perturbations $W_1(x_1, x_2)$ and $W_2(x_1, x_2)$ for orthotropic solids are written as [2]

$$\begin{gathered} a_{11}W_{1,11} + G_{12}W_{1,22} + (a_{12} + G_{12})W_{2,12} = 0, \\ (a_{12} + G_{12})W_{1,21} + (G_{12} - p)W_{2,11} + a_{22}W_{2,22} = 0, \end{gathered} \qquad (1)$$

where a_{11}, $a_{12} = a_{21}$, a_{22}, and G_{12} are the elastic coefficients; differentiation is denoted by subscripts after a comma.

System (1) should be supplemented by the boundary conditions on the free surface ($x_2 = 0$)

$$\sigma_{22}(x_1, 0) = 0, \qquad \sigma_{21}(x_1, 0) = 0 \qquad (2)$$

and the elastic relations

$$\sigma_{11} = a_{11}W_{1,1} + a_{12}W_{2,2}, \quad \sigma_{22} = a_{21}W_{1,1} + a_{22}W_{2,2}, \quad \sigma_{12} = \sigma_{21} = G_{12}(W_{1,2} + W_{2,1}).$$

The local buckling of the free surface is characterized by the fact that the displacement perturbations W_1 and W_2 should damp with distance from the epicenter of the perturbations over the surface (for $x_1 \to \pm\infty$) and into the depth from the surface (as $x_2 \to -\infty$).

We apply the Fourier transformation over the coordinate x_1 to the displacement perturbations:

$$U_j(\xi, x_2) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} W_j(x_1, x_2) \exp(i\xi x_1)\, dx_1 \qquad (j = 1, 2).$$

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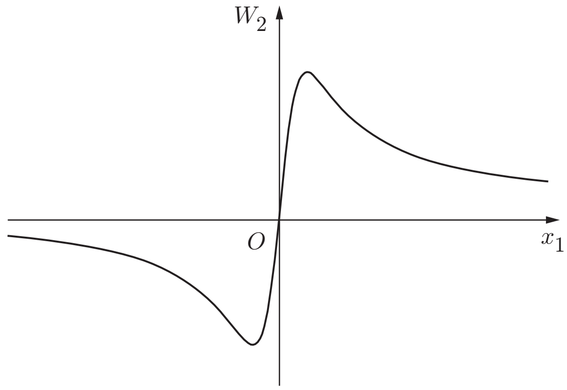

Fig. 1.

As a result of the Fourier transformation, system (1) becomes

$$\begin{gathered} -\xi^2 a_{11}U_1 + G_{12}U_{1,22} - i\xi(a_{12}+G_{12})U_{2,2} = 0, \\ -i\xi(a_{12}+G_{12})U_{1,2} - \xi^2(G_{12}-p)U_2 + a_{22}U_{2,22} = 0. \end{gathered} \tag{3}$$

The stress perturbations σ_{22} and σ_{21} in boundary conditions (2) are expressed in terms of the displacement perturbations W_1 and W_2 and, applying the Fourier transformation to the boundary conditions, we obtain

$$a_{22}U_{2,2} - i\xi a_{21}U_1 = 0, \qquad U_{1,2} - i\xi U_2 = 0. \tag{4}$$

System (3) is reduced to one equation for the function $U_1(\xi, x_2)$:

$$U_{1,2222} - 2\xi^2 aU_{1,22} + \xi^4 bU_1 = 0, \tag{5}$$

where $a = (a_{11}a_{22} - (a_{12}+G_{12})^2 + G_{12}(G_{12}-p))/(2a_{22}G_{12})$; $b = a_{11}(G_{12}-p)/(a_{22}G_{12})$.

The displacement perturbations W_1 and W_2 should damp with distance from the free surface $x_2 = 0$. The displacement perturbations U_1 and U_2 should possess the same property. Therefore, the solution of Eq. (5), which damps as $x_2 \to -\infty$, has the form

$$U_1(\xi, x_2) = \begin{cases} C_1 \exp(\xi(k_1x_2+\gamma)) + C_2 \exp(\xi(k_2x_2+\gamma)), & \xi \geqslant 0, \\ C_1 \exp(-\xi(k_1x_2+\gamma)) + C_2 \exp(-\xi(k_2x_2+\gamma)), & \xi < 0, \end{cases} \tag{6}$$

where C_1 and C_2 are arbitrary constants and γ is a constant ($\gamma < 0$); $k_{1,2} = \sqrt{a \pm \sqrt{a^2 - b}}$.

From system (3) we find $U_2(\xi, x_2)$, which damps as $x_2 \to -\infty$:

$$U_2(\xi, x_2) = i\begin{cases} C_1d_1 \exp(\xi(k_1x_2+\gamma)) + C_2d_2 \exp(\xi(k_2x_2+\gamma)), & \xi \geqslant 0, \\ -C_1d_1 \exp(-\xi(k_1x_2+\gamma)) - C_2d_2 \exp(-\xi(k_2x_2+\gamma)), & \xi < 0, \end{cases} \tag{7}$$

where $d_1 = k_1(a_1 - a_2k_1^2)$, $d_2 = k_2(a_1 - a_2k_2^2)$, $a_1 = (a_{11}a_{22} - (a_{12}+G_{12})^2)/a_3$, $a_2 = a_{22}G_{12}/a_3$, and $a_3 = (a_{12}+G_{12})(G_{12}-p)$.

The images of the displacement perturbations (6) and (7) correspond to the originals of the displacement perturbations

$$\begin{gathered} W_1(x_1, x_2) = -\sqrt{\frac{2}{\pi}}\Big(\frac{C_1(k_1x_2+\gamma)}{x_1^2+(k_1x_2+\gamma)^2} + \frac{C_2(k_2x_2+\gamma)}{x_1^2+(k_2x_2+\gamma)^2}\Big), \\ W_2(x_1, x_2) = \sqrt{\frac{2}{\pi}}\Big(\frac{C_1d_1x_1}{x_1^2+(k_1x_2+\gamma)^2} + \frac{C_2d_2x_1}{x_1^2+(k_2x_2+\gamma)^2}\Big). \end{gathered} \tag{8}$$

Figure 1 shows the transverse displacement $W_2(x_1, 0)$ of the free surface.

Substituting solutions (6) and (7) into boundary conditions (4), we obtain a homogeneous system of linear algebraic equations for arbitrary constants C_1 and C_2 (for $\xi \geqslant 0$ and $\xi < 0$, the systems coincide).

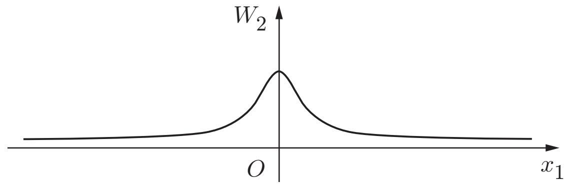

Fig. 2.

From the condition of existence of nontrivial solutions of the system, we obtain the following characteristic equation for the critical compressive load p_*:

$$(k_1 + d_1)(a_{22}d_2k_2 - a_{21}) - (k_2 + d_2)(a_{22}d_1k_1 - a_{21}) = 0. \tag{9}$$

From Eq. (9), we obtain the critical compressive load

$$\begin{gathered} p_1 = \Big(\sqrt{1 + \frac{4a_{11}a_{22}G_{12}^2}{(a_{11}a_{22} - a_{12}^2)^2}} - 1\Big)\frac{(a_{11}a_{22} - a_{12}^2)^2}{2a_{11}a_{22}G_{12}}, \\ p_2 = (2(a_{12} + G_{12})\sqrt{a_{11}a_{22}} - a_{11}a_{22} - 2a_{12}G_{12} - a_{12}^2)/G_{12}. \end{gathered} \tag{10}$$

The smaller of these positive roots gives the critical value p_*.

Reducing system (3) to one equation for the function $U_2(\xi, x_2)$

$$U_{2,2222} - 2\xi^2 aU_{2,22} + \xi^4 bU_2 = 0,$$

we obtain another form of surface buckling. By analogy with (6) and (7), we have

$$\begin{gathered} U_1(\xi, x_2) = i\begin{cases} C_1g_1 \exp(\xi(k_1x_2 + \gamma)) + C_2g_2 \exp(\xi(k_2x_2 + \gamma)), & \xi \geqslant 0, \\ -C_1g_1 \exp(-\xi(k_1x_2 + \gamma)) - C_2g_2 \exp(-\xi(k_2x_2 + \gamma)), & \xi < 0, \end{cases} \\ U_2(\xi, x_2) = \begin{cases} C_1 \exp(\xi(k_1x_2 + \gamma)) + C_2 \exp(\xi(k_2x_2 + \gamma)), & \xi \geqslant 0, \\ C_1 \exp(-\xi(k_1x_2 + \gamma)) + C_2 \exp(-\xi(k_2x_2 + \gamma)), & \xi < 0, \end{cases} \end{gathered} \tag{11}$$

where $g_1 = k_1(b_1 - b_2k_1^2)$, $g_2 = k_2(b_1 - b_2k_2^2)$, $b_1 = (G_{12}(G_{12} - p) - (a_{12} + G_{12})^2)/b_3$, $b_2 = a_{22}G_{12}/b_3$, and $b_3 = a_{11}(a_{12} + G_{12})$.

The images of the displacement perturbations (11) correspond to the originals of the displacement perturbations

$$\begin{gathered} W_1(x_1, x_2) = \sqrt{\frac{2}{\pi}}\Big(\frac{C_1g_1x_1}{x_1^2 + (k_1x_2 + \gamma)^2} + \frac{C_2g_2x_1}{x_1^2 + (k_2x_2 + \gamma)^2}\Big), \\ W_2(x_1, x_2) = -\sqrt{\frac{2}{\pi}}\Big(\frac{C_1(k_1x_2 + \gamma)}{x_1^2 + (k_1x_2 + \gamma)^2} + \frac{C_2(k_2x_2 + \gamma)}{x_1^2 + (k_2x_2 + \gamma)^2}\Big). \end{gathered} \tag{12}$$

For this form of surface buckling (Fig. 2), the characteristic equation is

$$(k_2g_2 - 1)(a_{22}k_1 + a_{21}g_1) - (k_1g_1 - 1)(a_{22}k_2 + a_{21}g_2) = 0. \tag{13}$$

The roots of Eq. (13) coincide with the roots of Eq. (10). Hence, one critical compressive loading correspond to two forms of local surface buckling (8) and (12). We note that the critical compressive load p_* depends only on the properties of the medium. Estimation of the critical load p_* shows that local surface buckling is possible not in all media. Thus, in an elastic isotropic medium, p_* corresponds to the load that exceeds the compressive strength limit for real materials. Hence, local surface buckling cannot occur in isotropic bodies at small subcritical strains.

In [2], it was shown that in an orthotropic medium with a low shear stiffness G_{12} there may be surface instability of the surface as a whole, since in this case the critical load p_* is lower than the compressive strength. The critical compressive load p_* obtained in [2] in studies of the surface instability of a generally free surface coincides with the critical load (10) in the case of local surface buckling.

Thus, in some media there may be local buckling of the surface of a half-plane under compression at small subcritical strains, and one critical load corresponds to two forms of local surface buckling.

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